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A COMPUTATIONALLY EFFICIENT ALGORITHM FOR THE SOLUTION OF EIGENPROBLEMS FOR LARGE STRUCTURES WITH NON-PROPORTIONAL DAMPING USING LANCZOS METHOD

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SUMMARY

In this paper, a solution method is presented to solve the eigenproblem arising in the dynamic analysis of non-proportional damping systems with symmetric matrices. The method is based on the Lanczos method to generate one pair of Krylov subspaces consisting of trial vectors, which is then used to reduce a large eigenvalue problem into a much smaller one. The method retains the n order quadratic eigenproblem, without employing the method of matrix augmentation traditionally used to cast the problem as a linear eigenproblem of order $2n$. In this process, the method preserves the sparseness and symmetry of the system matrices and does not invoke complex arithmetic; thus making it very economical for use in solving large problems. Numerical results are presented to demonstrate the efficiency and accuracy of the proposed method. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: eigenanalysis; non-proportional damping system; Krylov sequence; Lanczos recursion

1. INTRODUCTION

In the dynamic analysis of structures, the eigenvalue problem of the system should be solved *a priori* in order to avoid a resonance or to define the natural vibration characteristics. In most analyses presently employed, proportional damping is assumed for lack of a more realistic representation. That is, the damping of the structure is assumed to be such that the free modes of vibration of the damped system are identical to those for the undamped structure.[1] In these assumptions, the processes[2,3] to calculate the real-valued eigenvalues and associated eigenvectors are low in cost and straightforward. However, in most real systems the damping is non-proportional.[4] Even when proportional damping is assumed for each sub-system in the analysis of soil–structure interaction systems, composite structures, etc., the resulting damping for the complete structure will be non-proportional. Typically, in structural control problems the

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damping forces arise from a combination of direct velocity feedback control and Coriolis forces. In such problems, a more realistic model for the damping forces is available and should be used to simulate the correct response or to define the resonance frequency.[1] There are $2n$ eigenvalues for the system with n degrees of freedom and these occur either in real pairs or in complex conjugate pairs, depending upon whether correspond to overdamped or underdamped modes.

The common practice for the solution of problems with damping is to reformulate the quadratic equation into a linear one by doubling the order of the system. Transformation method such as QR,[5] LZ[6] or Jacobi[7] determine all the eigenvalues and the associated eigenvectors in arbitrary sequence. This is not very efficient in situations where only the lowest frequencies are of interest and there are a large number of degrees of freedom. Also transformation methods by their nature modify the initial matrices during the solution process and cannot fully take advantage of the sparseness of the matrices.[8]

Gupta[9,10] proposed a solution procedure based on a combined Sturm sequence and inverse iteration scheme for the linearized eigenproblem solution of spinning structures. Huseyin,[11] Plaut[12] and Janssens[13] discussed various ways of transforming the gyroscopic eigenproblem into a linear eigenproblem. Wittrick and Williams[14] used the spectrum slicing technique to find eigenvalues of spinning systems. Utku and Clemente[15] presented the inverse iteration method of the linearized eigenproblem of spinning structures. These procedures preserve the banded nature of the associated individual matrices, and are well suited for finding those frequencies within a certain range of interest. However, these algorithms involve a large number of complex operations. Despite recent improvements,[16] the techniques require increased storage capabilities, as well as high computation costs for matrices with a large bandwidth.[17]

Olson and Vandini,[18] Rodrigues and Gumer[17] and Leung[19] have presented the subspace iteration method based on simultaneous inverse iteration with Rayleigh–Ritz as a more efficient alternative algorithm than the inverse iteration procedure. These methods employed n order submatrices of the augmented linear eigenproblem in the iteration process by finding the block-partitioned nature of matrices of the linearized problem. It yields all the required modes simultaneously and does not suffer from the drawback of the high modes being less accurate than the lower modes because it avoids the round-off errors of the inverse iteration method by the deflection process. However, as in the inverse iteration method, a great number of complex manipulations are required in the iteration process.

The Lanczos method[20] was originally developed to evaluate the eigensolution of matrices through Rayleigh–Ritz reduction of the eigensystem to tridiagonal form. In his original paper, Lanczos described the algorithm as a method to evaluate the eigenvalues and the corresponding eigenvectors of a general matrix. The eigenvectors are obtained by a linear combination of a set of vectors, known as Lanczos vectors, computed in the course of the algorithm. Intensive researches[21–25] in the past twenty years have solved a number of difficulties concerning the stability of the Lanczos process and the algorithm has been extended to symmetric generalized eigenproblems. It is now widely accepted as the method for determining of eigenpairs of large sparse problems.[1]

More recently, the Lanczos algorithm was applied to systems with non-proportional damping. Borri and Mantegazza[26] and Bauchau[27] presented modifications of the Lanczos recursion procedure for spinning structures. Kim and Craig[28] and Gupta and Lawson[29,30] discussed block Lanczos algorithms for the linearized eigenproblem. Chen and Taylor[31] and Nour-Omid and Regelbrugge[1] proposed the Lanczos method for the damped systems with symmetric matrices using the linearized form of the quadratic eigenproblem. Rajakumar[32] presented the Lanczos

generalized two-sided recursion scheme to solve quadratic eigenproblems with unsymmetric matrices. The method retains the n order quadratic problem without the method of matrix augmentation traditionally used to cast the problem as a linear eigenproblem of order $2n$.

The quadratic eigenproblem of non-proportional damping systems with symmetric matrices can be solved by using the one-sided Lanczos algorithm once pointed out by Rajakumar.[3,2] In this paper, the one-sided Lanczos recursion scheme is proposed for non-proportionally damped systems with symmetric matrices, resulting in the reduction of storage space and solution time. The proposed method is based on the Rajakumar's algorithm,[3,2] which also retains the n order quadratic eigenproblem and extracts the Lanczos vectors in real domain. In the following section, the basic concepts of the proposed method are presented. In Section 3, the efficiency and accuracy of the proposed method are shown for various numerical examples.

2. PROPOSED METHOD

The quadratic eigenproblem of the damped structure frequently encountered in various engineering fields can be expressed as

$$\lambda_i^2 M\phi_i + \lambda_i C\phi_i + K\phi_i = 0, \quad i = 1, \ldots, n \tag{1}$$

where M, C and K are symmetric mass, damping and stiffness matrices of order n, respectively. The mass matrix M is assumed to be positive semi-definite and the stiffness matrix K positive definite. λ_i and ϕ_i are the ith eigenvalue and associated eigenvector of the system, and both are complex valued in general. The eigenvectors satisfy the orthogonal condition[3,2]

$$\phi_j^{\mathrm{T}} M\tilde{\phi}_i + \tilde{\phi}_j^{\mathrm{T}} M\phi_i + \phi_j^{\mathrm{T}} C\phi_i = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \tag{2}$$

with

$$\tilde{\phi}_i = \lambda_i \phi_i. \tag{3}$$

$\tilde{\phi}_i$ is obtainable from the independent eigenvector ϕ_i and, therefore, called the dependent eigenvector.

2.1. Krylov sequence

The Lanczos algorithm is equivalent to obtaining the Rayleigh–Ritz approximation with the vectors in the Krylov sequence as the trial vectors.[3] One pair of the Krylov sequence of vectors that will map the independent and dependent eigenvectors ϕ_i and $\tilde{\phi}_i$, respectively, can be easily obtained by considering the linearized form of the quadratic eigenproblem

$$\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \phi_i \\ \tilde{\phi}_i \end{Bmatrix} = \lambda_i \begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{Bmatrix} \phi_i \\ \tilde{\phi}_i \end{Bmatrix} \tag{4}$$

Given one set of the arbitrary starting vectors as $(x_1\ y_1)$, we can get the Krylov sequence, used to obtain the best approximation to the wanted eigenvectors ϕ_i by

$$x_{i+1} = -K^{-1}(Cx_i + My_i) \quad \text{for } i = 1, 2, \ldots \tag{5}$$

and $\tilde{\phi}_i$ by

$$y_{i+1} = x_i \quad \text{for } i = 1, 2, \ldots \tag{6}$$

2.2. Lanczos recursion

The Lanczos method involves supplementing the Krylov sequence with Gram–Schmidt orthogonalization process at each step.[3] In this section, we present the facts that the orthogonalization is required with respect to only two preceding vectors, and one pair of the resulting Lanczos vectors of the Krylov sequence transforms a quadratic eigenproblem into a standard eigenproblem with a tridiagonal matrix.

To derive the Lanczos algorithm, assume that the first m independent and dependent Lanczos vectors, $\{v_1, v_2, \ldots, v_m\}$ and $\{p_1, p_2, \ldots, p_m\}$, are found and the next independent and dependent Lanczos vectors, v_{m+1} and p_{m+1}, are to be constructed. v_{m+1} and p_{m+1} are obtained by first computing preliminary vectors $\hat{v}_{m+1}$ and $\hat{p}_{m+1}$ as in the Krylov sequence

$$\hat{v}_{m+1} = -K^{-1}(Cv_m + Mp_m) \tag{7}$$

$$\hat{p}_{m+1} = v_m \tag{8}$$

In general, these preliminary vectors can be expressed as a linear combination of all the previous Lanczos vectors and a residual vector as

$$\hat{v}_{m+1} = \bar{v}_{m+1} + \alpha_m v_m + \beta_m v_{m-1} + \varepsilon_m v_{m-2} + \cdots \tag{9}$$

$$\hat{p}_{m+1} = \bar{p}_{m+1} + \alpha_m p_m + \beta_m p_{m-1} + \varepsilon_m p_{m-2} + \cdots \tag{10}$$

with

$$\bar{v}_{m+1} = \gamma_{m+1} v_{m+1} \tag{11}$$

$$\bar{p}_{m+1} = \gamma_{m+1} p_{m+1} \tag{12}$$

where the residual vectors, $\bar{v}_{m+1}$ and $\bar{p}_{m+1}$, are the pure components of $\hat{v}_{m+1}$ and $\hat{p}_{m+1}$ orthogonal to all the previous Lanczos vectors, and γ_{m+1} is the pseudo length of $\bar{v}_{m+1}$ and $\bar{p}_{m+1}$. $\alpha_m, \beta_m, \varepsilon_m, \ldots$ are the components of the previous Lanczos vectors contained in $\hat{v}_{m+1}$ and $\hat{p}_{m+1}$, respectively. The new Lanczos vectors satisfy the following orthogonal conditions for all the previous independent and dependent Lanczos vectors:

$$v_i^{\mathrm{T}} M p_{m+1} + p_i^{\mathrm{T}} M v_{m+1} + v_i^{\mathrm{T}} C v_{m+1} = 0 \quad \text{for } i = 1, \ldots, m \tag{13}$$

$$v_{m+1}^{\mathrm{T}} M p_{m+1} + p_{m+1}^{\mathrm{T}} M v_{m+1} + v_{m+1}^{\mathrm{T}} C v_{m+1} = \delta_{m+1} \tag{14}$$

where

$$\delta_{m+1} = \mathrm{sign}(\bar{v}_{m+1}^{\mathrm{T}} M \bar{p}_{m+1} + \bar{p}_{m+1}^{\mathrm{T}} M \bar{v}_{m+1} + \bar{v}_{m+1}^{\mathrm{T}} C \bar{v}_{m+1}) \tag{15}$$

Here, the pseudo length of the new Lanczos vectors is normalized to be 1 or -1.

These component coefficients can be evaluated by the orthogonal conditions, equations (13) and (14), for every Lanczos vectors. To obtain the coefficient α_m, we premultiply both sides of equations (9) and (10) by $(p_m^{\mathrm{T}} M + v_m^{\mathrm{T}} C)$ and $v_m^{\mathrm{T}} M$, respectively, and add together. Then using the orthogonal relationships, we can get the following result:

$$\alpha_m = (v_m^{\mathrm{T}} M \hat{p}_{m+1} + p_m^{\mathrm{T}} M \hat{v}_{m+1} + v_m^{\mathrm{T}} C \hat{v}_{m+1}) \delta_m \tag{16}$$

Substituting equations (7) and (8) into $\hat{v}_{m+1}$ and $\hat{p}_{m+1}$ in equation (16), respectively, we can readily obtain the coefficient α_m

$$\alpha_m = [\,-(p_m^{\mathrm{T}}M + v_m^{\mathrm{T}}C)K^{-1}(Cv_m + Mp_m) + v_m^{\mathrm{T}}Mv_m]\delta_m \tag{17}$$

The component β_m may be obtained similarly by premultiplying $(p_{m-1}^{\mathrm{T}}M + v_{m-1}^{\mathrm{T}}C)$ and $v_{m-1}^{\mathrm{T}}M$, respectively, and adding together. Then using the orthogonal conditions, we can have

$$\beta_m = (v_{m-1}^{\mathrm{T}}M\hat{p}_{m+1} + p_{m-1}^{\mathrm{T}}M\hat{v}_{m+1} + v_{m-1}^{\mathrm{T}}C\hat{v}_{m+1})\delta_{m-1} \tag{18}$$

Using equations (7) and (8) to eliminate $\hat{v}_{m+1}$ and $\hat{p}_{m+1}$ in equation (18) gives

$$\beta_m = [\,-(p_{m-1}^{\mathrm{T}}M + v_{m-1}^{\mathrm{T}}C)K^{-1}(Cv_m + Mp_m) + v_{m-1}^{\mathrm{T}}Mv_m]\delta_{m-1} \tag{19}$$

and applying the transpose of equations (7) and (8) to $-(p_{m-1}^{\mathrm{T}}M + v_{m-1}^{\mathrm{T}}C)K^{-1}$ and $v_{m-1}^{\mathrm{T}}M$, we can get

$$\beta_m = (\hat{v}_m^{\mathrm{T}}Mp_m + \hat{p}_m^{\mathrm{T}}Mv_m + \hat{v}_m^{\mathrm{T}}Cv_m)\delta_{m-1} \tag{20}$$

Finally, expanding $\hat{v}_m$ and $\hat{p}_m$ in terms of equations (9) and (10), respectively, and then using the orthogonality relationships given in equations (13) and (14), we can get the following result:

$$\beta_m = (\bar{v}_m^{\mathrm{T}}Mp_m + \bar{p}_m^{\mathrm{T}}Mv_m + \bar{v}_m^{\mathrm{T}}Cv_m)\delta_{m-1} \tag{21}$$

or rewriting β_m for the $(m+1)$st vector

$$\beta_{m+1} = (\bar{v}_{m+1}^{\mathrm{T}}Mp_{m+1} + \bar{p}_{m+1}^{\mathrm{T}}Mv_{m+1} + \bar{v}_{m+1}^{\mathrm{T}}Cv_{m+1})\delta_m \tag{22}$$

The new independent and dependent Lanczos vectors are obtained simply by scaling the pure vectors, $\bar{v}_{m+1}$ and $\bar{p}_{m+1}$, as

$$v_{m+1} = \bar{v}_{m+1}/\gamma_{m+1} \tag{23}$$

$$p_{m+1} = \bar{p}_{m+1}/\gamma_{m+1} \tag{24}$$

where γ_{m+1} is the pseudo length of $\bar{v}_{m+1}$ and $\bar{p}_{m+1}$. Therefore, using the expressions for v_{m+1} and p_{m+1} in equation (22), we can obtain β_{m+1} and γ_{m+1}

$$\begin{aligned}\Delta_{m+1} &= \beta_{m+1}\gamma_{m+1} \\ &= (\bar{v}_{m+1}^{\mathrm{T}}M\bar{p}_{m+1} + \bar{p}_{m+1}^{\mathrm{T}}M\bar{v}_{m+1} + \bar{v}_{m+1}^{\mathrm{T}}C\bar{v}_{m+1})\delta_m\end{aligned} \tag{25}$$

$$\gamma_{m+1} = |\Delta_{m+1}|^{1/2} \tag{26}$$

$$\delta_{m+1} = \mathrm{sign}(\Delta_{m+1}) \tag{27}$$

$$\beta_{m+1} = |\Delta_{m+1}|^{1/2}\delta_{m+1} \tag{28}$$

If we apply the same procedures above to equation (21), we can get the coefficient ε_m by

$$\varepsilon_m = (\bar{v}_{m-1}^{\mathrm{T}}Mp_m + \bar{p}_{m-1}^{\mathrm{T}}Mv_m + \bar{v}_{m-1}^{\mathrm{T}}Cv_m)\delta_{m-2} \tag{29}$$

Substituting equations (11) and (12) for $\bar{v}_{m-1}$ and $\bar{p}_{m-1}$ in equation (29), respectively, we can obtain

$$\begin{aligned}\varepsilon_m &= \gamma_{m-1}(v_{m-1}^{\mathrm{T}}Mp_m + p_{m-1}^{\mathrm{T}}Mv_m + v_{m-1}^{\mathrm{T}}Cv_m)\delta_{m-2} \\ &= 0\end{aligned} \tag{30}$$

Since all terms on the right-hand side of equation (30) vanish due to the orthogonality conditions, we can get the result that $\varepsilon_m = 0$. A similar procedure can be used to show that all further terms in the expansions for $\hat{v}_{m+1}$ and $\hat{p}_{m+1}$, equations (9) and (10), are zero.

Therefore, equations (9) and (10) can be rewritten as the three-term recurrence formulas for derivation of the pure components, $\bar{v}_{m+1}$ and $\bar{p}_{m+1}$, as

$$\bar{v}_{m+1} = -K^{-1}(Cv_m + Mp_m) - \alpha_m v_m - \beta_m v_{m-1} \tag{31}$$

$$\bar{p}_{m+1} = v_m - \alpha_m p_m - \beta_m p_{m-1} \tag{32}$$

The proposed method is the Lanczos recursion scheme for the quadratic eigenproblem of damped systems without the reformulation of the augmented eigenproblem. The summary of the proposed Lanczos algorithm is presented in Table I. If arbitrary random vectors $\bar{v}_1$ and $\bar{p}_1$ are taken in real value, the recursive procedure to generate the Lanczos vectors can be operated in real domain. It is very significant because most efforts of solution process are spent on generating the Lanczos vectors.

2.3. Reduction to tridiagonal system

The quadratic eigenproblem in equation (1) can be rewritten as

$$K\phi_i = -\lambda_i(C\phi_i + M\tilde{\phi}_i) \tag{33}$$

with

$$\tilde{\phi}_i = \lambda_i \phi_i \tag{34}$$

Table I. Proposed Lanczos algorithm

Given arbitrary random vectors $\bar{v}_1$ and $\bar{p}_1$, then
Set

$\Delta_1 = \lfloor \bar{v}_1^{\mathrm{T}} M \bar{p}_1 + \bar{p}_1^{\mathrm{T}} M \bar{v}_1 + \bar{v}_1^{\mathrm{T}} C \bar{v}_1 \rfloor$

$\gamma_1 = |\Delta_1|^{1/2}$

$\delta_1 = \mathrm{sign}(\Delta_1)$

$v_1 = \bar{v}_1/\gamma_1$

$p_1 = \bar{p}_1/\gamma_1$

For $i = 1, 2, 3, \ldots, m$ $(m \leqslant 2n)$, repeat

$\hat{v}_{i+1} = -K^{-1}(Cv_i + Mp_i)$

$\hat{p}_{i+1} = v_i$

$\alpha_i = \lfloor -(p_i^{\mathrm{T}} M + v_i^{\mathrm{T}} C) K^{-1}(Cv_i + Mp_i) + v_i^{\mathrm{T}} M v_i \rfloor \delta_i$

$\bar{v}_{i+1} = \hat{v}_{i+1} - \alpha_i v_i - \beta_i v_{i-1}$

$\bar{p}_{i+1} = \hat{p}_{i+1} - \alpha_i p_i - \beta_i p_{i-1}$

$\Delta_{i+1} = \lfloor \bar{v}_{i+1}^{\mathrm{T}} M \bar{p}_{i+1} + \bar{p}_{i+1}^{\mathrm{T}} M \bar{v}_{i+1} + \bar{v}_{i+1}^{\mathrm{T}} C \bar{v}_{i+1} \rfloor \delta_i$

$\gamma_{i+1} = |\Delta_{i+1}|^{1/2}$

$\delta_{i+1} = \mathrm{sign}(\Delta_{i+1})$

$\beta_{i+1} = |\Delta_{i+1}|^{1/2} \delta_{i+1}$

After m steps, we have two sets of the independent and dependent Lanczos vectors, $V = [v_1\ v_2, \ldots, v_m]$ and $P = [p_1\ p_2\ \cdots\ p_m]$. Here, we use the Rayleigh–Ritz method to obtain the reduced one from the quadratic eigenproblem as

$$\phi_i = V\psi_i \tag{35}$$

$$\tilde{\phi}_i = P\psi_i \tag{36}$$

Substituting equations (35) and (36) into equations (33) and (34), multiplying the results by $(V^{\mathrm{T}}C + P^{\mathrm{T}}M)K^{-1}$ and $V^{\mathrm{T}}M$, respectively, and then adding the two equations, we can obtain the tridiagonalized standard eigenproblem of order $m \ll 2n$

$$\Delta\psi_i = \lambda_i \Delta T \psi_i \quad \text{for } i = 1, \ldots, m \tag{37}$$

or

$$T\psi_i = \theta_i \psi_i \tag{38}$$

where

$$\begin{aligned} \Delta &= V^{\mathrm{T}}MP + P^{\mathrm{T}}MV + V^{\mathrm{T}}CV \\ &= \mathrm{diag}\,\{\delta_1, \delta_2, \ldots, \delta_m\} \end{aligned} \tag{39}$$

$$\begin{aligned} T &= -(V^{\mathrm{T}}C + P^{\mathrm{T}}M)K^{-1}(CV + MP) + V^{\mathrm{T}}MV \\ &= \begin{bmatrix} \alpha_1 & \beta_2 & & & & \\ \gamma_2 & \alpha_2 & \beta_3 & & & \\ & \gamma_3 & \alpha_3 & \beta_4 & & \\ & & \gamma_4 & \ddots & \ddots & \\ & & & \ddots & \ddots & \beta_m \\ & & & & \gamma_m & \alpha_m \end{bmatrix} \end{aligned} \tag{40}$$

$$\theta_i = 1/\lambda_i \tag{41}$$

We solve the reduced eigenproblem, equation (38), by the QR algorithm, which is readily available in the IMSL library. Because the matrix T of the reduced eigenproblem is unsymmetric, a solution process of the reduced eigenproblem should be operated in complex domain. However, much less solution time is required for solving the reduced eigenproblem because the matrix T of the reduced eigenproblem is tridiagnoarized and order of the matrix T is reduced in $m \ll 2n$. The solutions of this reduced eigenproblem relate to those of the original eigenproblem through equations (35), (36) and (41).

2.4. Residual vector

Since the Rayleigh–Ritz method is applied to the algorithm, the solutions, $\lambda_i^{(m)}$ and $\phi_i^{(m)}$, resulting from the reduced eigenproblem of order m are generally approximations to the least dominant eigenpairs of the original quadratic eigenproblem. To show the accuracy of these approximations, the residual vector is defined as

$$r_i^{(m)} = K^{-1}M\tilde{\phi}_i^{(m)} + K^{-1}C\phi_i^{(m)} + \frac{1}{\lambda_i^{(m)}}\phi_i^{(m)}. \tag{42}$$

This residual vector can be obtained easily by the three-term recurrence formula, equation (31). Matrix form of equation (31) is

$$\gamma_{m+1} V_{m+1} e_m^{\mathrm{T}} = -K^{-1}(CV + MP) - V^{\mathrm{T}} T \tag{43}$$

where $e_m^{\mathrm{T}} = \langle 0\ 0\ \cdots\ 0\ 1 \rangle$ is the mth column of a $m \times m$ unit matrix. Postmultiplying equation (43) by ψ_i, and then taking $T\psi_i = (1/\lambda_i)\psi_i$, we can obtain

$$K^{-1} M \tilde{\phi}_i^{(m)} + K^{-1} C \phi_i^{(m)} + \frac{1}{\lambda_i^{(m)}} \phi_i^{(m)} = -\gamma_{m+1} v_{m+1} \psi_i(m) \tag{44}$$

where $\psi_i(m)$ represents the mth element of the vector ψ_i. Therefore, the residual vector, $r_i^{(m)}$, can be found simply as

$$r_i^{(m)} = -\gamma_{m+1} v_{m+1} \psi_i(m) \tag{45}$$

Since $r_i^{(m)} = 0$ corresponds to an exact solution, we can assess the accuracy of the approximate solution by examining the magnitude of the residual norm $|r_i^{(m)} \cdot r_i^{(m)}|^{1/2}$.

However, Bathe[33] proposed that the following physical error norm should be used as measure the accuracy of the approximations

$$e_i = \frac{\| [(\lambda_i^{(m)})^2 M + \lambda_i^{(m)} C + K] \phi_i^{(m)} \|_2}{\| K \phi_i^{(m)} \|_2} < 10^{-6} \tag{46}$$

which yields a stable eigensolution and sufficient accuracy in the calculated eigenvalues and eigenvectors for practical analysis. In this paper, the norm of the residual vector is used in the preliminary check of the convergence and the physical error norm is used for the final check.

2.5. Loss of orthogonality

The Lanczos algorithm presented here, involving orthogonalization with only the two preceding vectors at each step, is subjected to loss of orthogonality with respect to earlier vectors due to round-off errors. If such errors are not corrected when they reach a critical size, the Lanczos vectors may become linearly dependent. A remedy to prevent the loss of orthogonality is to use the double Gram–Schmidt orthogonalization process at each step.[34] In some cases, selectively applying reorthogonalization may be sufficient.[21,24,25]

In this paper, the double Gram–Schmidt orthogonalization process is used to retain the orthogonality of the Lanczos vectors as

$$v_{m+1} = v_{m+1} - \sum_{i=1}^{m-2} \sum_{i=1}^{m-2} \eta_i v_i \tag{47}$$

$$p_{m+1} = p_{m+1} - \sum_{i=1}^{m-2} \sum_{i=1}^{m-2} \eta_i p_i \tag{48}$$

where

$$\eta_i = (v_i^{\mathrm{T}} M p_{m+1} + p_i^{\mathrm{T}} M v_{m+1} + v_i^{\mathrm{T}} C v_{m+1}) \delta_i \tag{49}$$

3. NUMERICAL EXAMPLES

In this section, three examples are presented to assess the effectiveness of the proposed method and the robustness of the developed computer program. We depict the number of eigenpairs within the physical error norm 10^{-6} and the variation of each physical error norm of the first ten eigenpairs as increasing the Lanczos vectors. Also, we compare the CPU time for the proposed method with those of Chen's[31] and Rajakumar's[32] algorithms. The former is the Lanczos algorithm dealing with the linearized eigenproblem with symmetric matrices, and the latter the two-sided recursion scheme treating the case of unsymmetric matrices without resorting to the augmented eigenproblem. All executions are done on the CONVEX with 100 MIPS and 200 MFLOPS. In the following examples the initial vectors of the proposed method, $\bar{v}_1$ and $\bar{p}_1$, are set equal to $K^{-1}\langle 1 \cdots 1\rangle^{\mathrm{T}}$.

3.1. Simple spring-damper system[34]

The finite element discretization of the system results in a diagonal mass matrix, a tridiagonal damping and stiffness matrices of the following forms:

$$M = mI \tag{50}$$

$$C = \alpha M + \beta K \tag{51}$$

$$K = k\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & 2 & -1 & \\ & & & -1 & 1 & \end{bmatrix} \tag{52}$$

where α and β are the damping coefficients of the Rayleigh damping. To compare the analytical solutions with the numerical results obtained from the proposed method, the Rayleigh damping matrix is assumed. The analytical solutions can be derived from the following relationships:

$$\lambda_{2i-1,2i} = -\xi_i\omega_i \pm j\omega_i\sqrt{1-\xi_i^2} \quad \text{for } i = 1, \ldots, n \tag{53}$$

$$\xi_i = \frac{1}{2}\left(\frac{\alpha}{\omega_i} + \beta\omega_i\right) \tag{54}$$

$$\omega_i = 2\sqrt{\frac{m}{k}}\sin\frac{2\mathrm{i}-1}{2n+1}\frac{\pi}{2} \tag{55}$$

where ω_i and ξ_i are the undamped natural frequency and modal damping ratio, respectively.

A system with order 50 is used in the analysis. k and m are 1, and the coefficients, α and β, of the Rayleigh damping are 0·01, respectively. The developed computer program is used to make 10 runs, generating 10, 20, …, 100 independent and dependent Lanczos vectors of the damped system. That is, 10 reduced problems of size 10, 20, …, 100 are produced and solved.

Figure 1 summarizes the number of acceptable Ritz pairs obtained during the ten runs. Here, we call a Ritz pair having the physical error norm within 10^{-6} acceptable. Figure 2 shows the

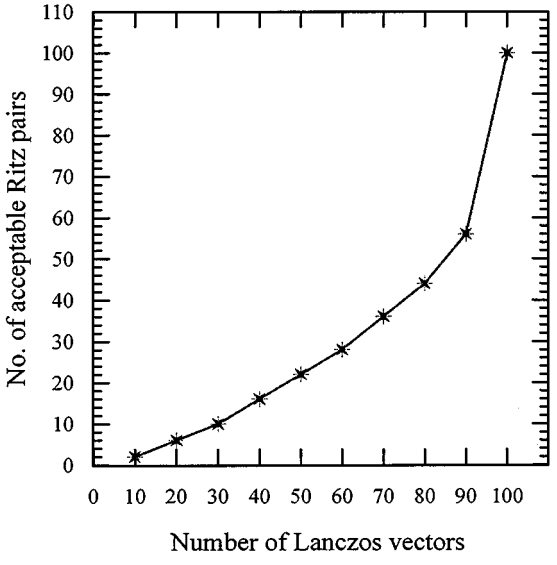

Figure 1. Number of acceptable Ritz pairs of simple system

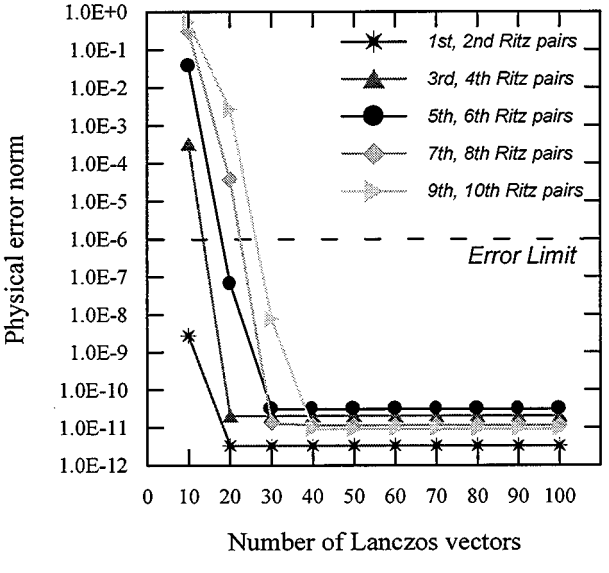

Figure 2. Variation of error norms of first ten Ritz pairs of simple system

variation of each physical error norm of the first ten Ritz pairs as increasing the Lanczos vectors. Note that the initial slope of the line in Figure 1 is about 1/3, which indicates that three independent and dependent Lanczos vectors are enough to produce an acceptable complex conjugate pair of the eigenvalues and associated eigenvectors being sought. This result implies

that the algorithm developed here is highly effective in finding the desired eigenvalues and eigenvectors of the damped system. In the last run, in which the whole problem is solved actually, all the Ritz values are compared with the analytical solutions. The eigenvalues by the two methods are exactly the same up to six decimal points. This verifies the correctness of the algorithm and more importantly the robustness of the developed program. The CPU time elapsed to solve each of the 10 problems is shown in Table II. We compare the results with those for implementations of Chen's[31] and Rajakumar's[32] methods. The solution times for the proposed method are less than those for the other methods.

3.2. Cantilever beam and lumped dampers

A cantilever beam with tangential viscous dampers is considered as a representative of soil–structure interaction systems. Figure 3 shows the geometric configuration of the system. Dimensionless value of Young's modulus for the beam material is taken as 1000, while mass density, section area and inertia are specified to be of unit value. The cantilever beam is divided into 100 equal elements and has 200 degrees of freedom. A consistent mass is used to define M. The damping matrix, C, consists of the Rayleigh damping and lumped dampers attached on each node, resulting in a non-proportional damping. The Rayleigh coefficients α and β in equation (51) are 0·001, and the damping value of each lumped damper is 0·3.

As in the previous example, the developed computer program is used to make 10 runs. The results are summarized in Figures 4, 5 and Table III.

Table II. CPU time (s) for solution of simple system

No. of Lanczos vectors	Proposed method (ratio)	Chen's method (ratio)	Rajakumar's method (ratio)
10	0·17(1·00)	1·81(10·65)	0·20(1·18)
20	0·44(1·00)	2·44(5·55)	0·60(1·36)
30	0·90(1·00)	3·42(3·80)	1·14(1·27)
40	1·52(1·00)	4·42(2·91)	1·91(1·26)
50	2·27(1·00)	5·75(2·53)	2·88(1·27)
60	3·16(1·00)	7·41(2·34)	4·01(1·27)
70	4·19(1·00)	9·31(1·90)	5·39(1·29)
80	5·73(1·00)	12·23(2·13)	7·40(1·29)
90	6·78(1·00)	13·93(2·05)	8·74(1·29)
100	8·31(1·00)	17·11(2·06)	10·65(1·28)

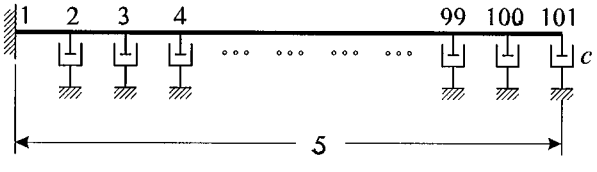

Figure 3. Cantilever beam with lumped dampers

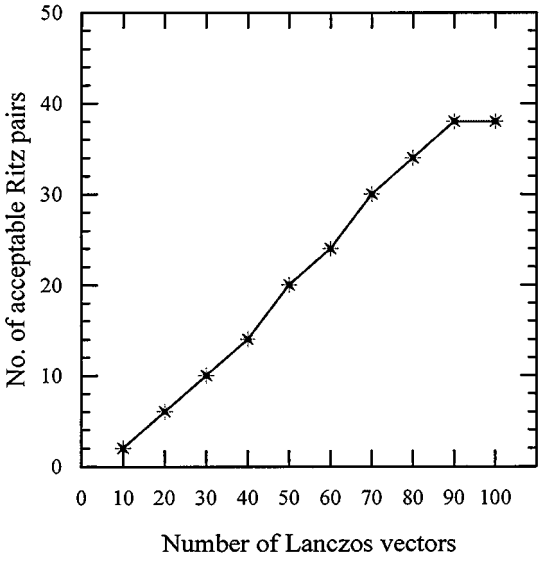

Figure 4. Number of acceptable Ritz pairs of cantilever beam

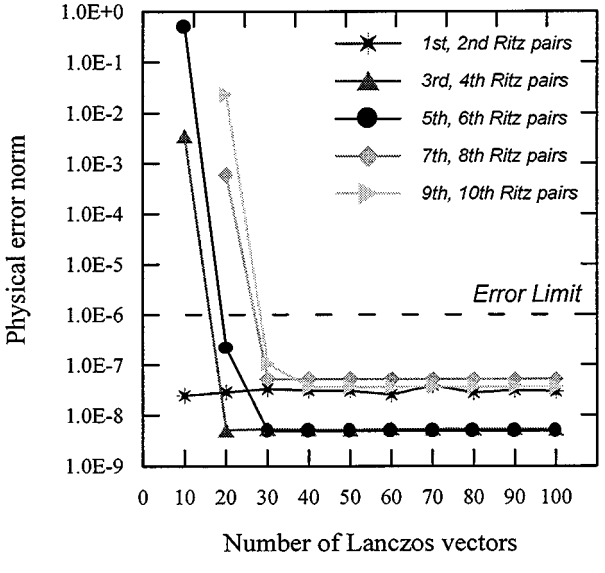

Figure 5. Variation of error norms of first ten Ritz pairs of cantilever beam

3.3. Three-dimensional frame structure with lumped dampers

A three-dimensional framed structure with concentrated dampers is presented. Two layers of the foundation are damped as shown in Figure 6. The model has 1008 degrees of freedom. The

Table III. CPU time (s) for solution of cantilever beam

No. of Lanczos vectors	Proposed method (ratio)	Chen's method (ratio)	Rajakumar's method (ratio)
10	0·75(1·00)	7·22(9·63)	1·02(1·36)
20	1·98(1·00)	13·25(6·69)	2·71(1·37)
30	3·58(1·00)	20·86(5·82)	4·93(1·38)
40	5·83(1·00)	29·26(5·02)	7·79(1·34)
50	8·26(1·00)	37·38(4·53)	11·26(1·36)
60	11·36(1·00)	42·87(3·77)	15·82(1·39)
70	14·88(1·00)	56·24(3·78)	20·65(1·39)
80	19·41(1·00)	69·43(3·58)	26·84(1·38)
90	23·53(1·00)	79·52(3·38)	33·25(1·41)
100	28·59(1·00)	93·49(3·27)	41·16(1·44)

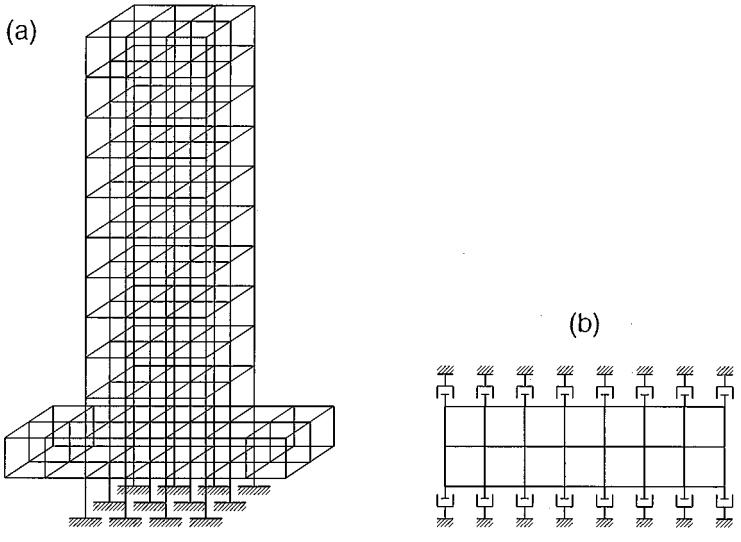

Figure 6. (a) Three-dimensional frame structure; (b) damping of two-layer foundation

material and cross section properties are $E = 2{\cdot}1 \times 10^{11}$, $I = 8{\cdot}3 \times 10^{-6}$, $A = 0{\cdot}01$ and $\rho = 7850$. The consistent mass matrix is used to define M. The damping matrix, C, consists of the Rayleigh damping and concentrated dampers, resulting in a non-proportional damping. The Rayleigh coefficients α is $-0{\cdot}92$ and $\beta = 0{\cdot}1016$. The damping value of each concentrated damper is 1000.

The results are shown in Figure 7, 8 and Table IV. It can be seen that the results are similar to those of the previous examples.

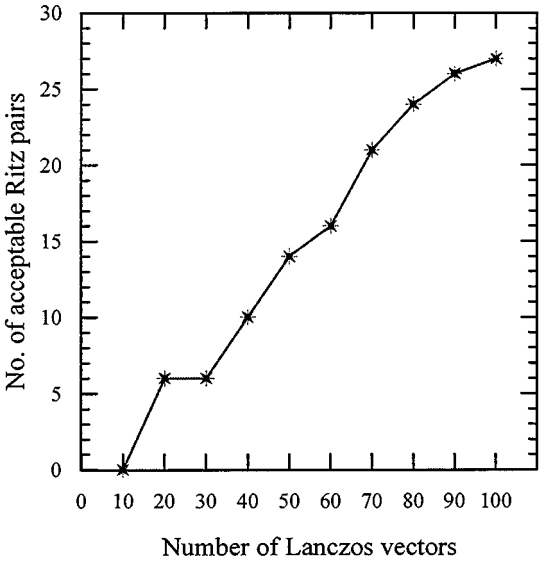

Figure 7. Number of acceptable Ritz pairs of frame structure

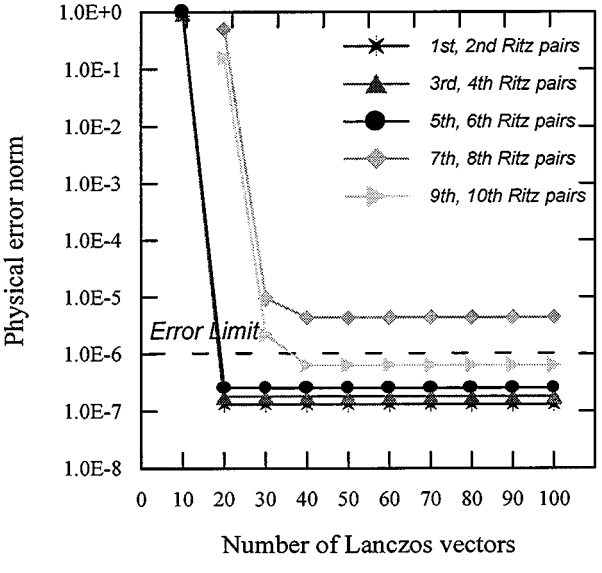

Figure 8. Variation of error norms of first ten Ritz pairs of frame structure

4. CONCLUSIONS

In this paper, the Lanczos method to solve the quadratic eigenproblem with symmetric matrices is presented. Since the matrices of the quadratic eigenproblem are retained in their original order

Table IV. CPU time (s) for solution of frame structure

No. of Lanczos vectors	Proposed method (ratio)	Chen's method (ratio)	Rajakumar's method (ratio)
10	34·06(1·00)	189·18(5·55)	48·42(1·42)
20	65·76(1·00)	376·79(5·73)	94·99(1·44)
30	99·70(1·00)	569·15(5·71)	144·58(1·45)
40	136·03(1·00)	768·68(5·65)	197·09(1·45)
50	173·10(1·00)	972·44(5·62)	268·01(1·55)
60	212·97(1·00)	1161·09(5·45)	322·99(1·52)
70	257·97(1·00)	1395·08(5·41)	388·92(1·51)
80	310·20(1·00)	1589·98(5·13)	455·56(1·47)
90	363·54(1·00)	1824·10(5·02)	525·63(1·45)
100	422·21(1·00)	2060·47(4·88)	604·98(1·43)

n, unlike the linearized problem with matrices of order $2n$ as used in Chen's[31] scheme, large size problems can be handled with efficiency. The proposed method fully takes advantage of the sparseness and symmetry of the system matrices and requires no complex arithmetic operations during the modal reduction process. It implies that the algorithm is very economical when compared with other algorithms such as Chen's[31] and Rajakumar's[32] methods. The numerical results from test problems prove that the proposed method is very accurate and robust. Finally, experiences show that if $3p$ independent and dependent Lanczos vectors are used in the reduction process, the p lowest eigenpairs computed satisfy the required accuracy.

ACKNOWLEDGEMENT

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